Multi-Partite Entanglement Inequalities via Spin Vector Geometry

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We introduce inequalities for multi-partite entanglement, derived from the geometry of spin vectors. The criteria are constructed iteratively from cross and dot products between the spins of individual subsystems, each of which may have arbitrary dimension. For qubit ensembles the maximum violation for our inequalities is larger than that for the Mermin-Klyshko Bell inequalities, and the maximally violating states are different from Greenberger-Horne-Zeilinger states. Our inequalities are violated by certain bound entangled states for which no Bell-type violation has yet been found.

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Entanglement is one of the most mysterious features of quantum physics and a key ingredient in the science of quantum information. While initial research was focussed on bipartite entanglement, multipartite entanglement has attracted increasing attention, because it was realized that multipartite entangled states can exhibit qualitatively different features [1]. Multi-partite entangled states are also important for most applications envisaged in quantum information, such as quantum computation [2]. Recently, multi-partite entanglement has been studied using multi-partite Bell inequalities [3, 4], the partial transposition criterion [6], and a variety of other methods [7, 8].

Here we develop a new approach based on the geometry of spin vectors. Our results are relevant for any system where operators analogous to spin can be defined and measured, e.g. for multi-mode light fields [9] and Bose-Einstein condensates [10]. In any finite-dimensional Hilbert space one can always define operators $\{J_x, J_y, J_z\} = \vec{J}$ that satisfy the commutation relations of angular momentum, $[J_u, J_v] = i\epsilon^{uvw} J_w$ for $u, v, w \in \{x, y, z\}$. The transformations $U(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{J}}$ generated by these operators, where $\vec{\alpha}$ is a numerical vector, form a representation of the group $SU(2)$. Applying $U(\vec{\alpha})$ to the quantum states leads to an $SO(3)$ rotation $R(\vec{\alpha})$ of the vector $\langle\vec{J}\rangle = \{\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle\}$ of spin expectation values. The vector $\vec{\alpha}$ gives both axis and angle of rotation.

Dot Product: Our inequalities involve the expectation values of operators that are constructed from the spins of subsystems via the cross and dot product. We will first illustrate the principle with the simplest example, the dot product between two spins of magnitude j_1 and j_2. For a product state $|\psi_{12}\rangle = |\phi_1\rangle|\chi_2\rangle$ the expectation value of the dot product is

$$\langle\psi_{12}|\vec{J}^{(1)}\cdot\vec{J}^{(2)}|\psi_{12}\rangle = \langle\phi_1|\vec{J}^{(1)}|\phi_1\rangle\cdot\langle\chi_2|\vec{J}^{(2)}|\chi_2\rangle, \quad (1)$$

which is the scalar product of two vectors $\langle\vec{J}^{(1)}\rangle = \langle\phi_1|\vec{J}^{(1)}|\phi_1\rangle$ and $\langle\vec{J}^{(2)}\rangle = \langle\chi_2|\vec{J}^{(2)}|\chi_2\rangle$. The modulus of the scalar product is bounded by the product of the norms of the two vectors. Furthermore the norm $||\langle\vec{J}^{(1)}\rangle||$ cannot exceed j_1. This can be seen by noting that by a rotation the vector $\langle\vec{J}^{(1)}\rangle$ can always be brought to a form where only one of its components, say $\langle J_z^{(1)}\rangle$, is different from zero, without changing its norm. One has $|\langle J_z^{(1)}\rangle| \leq j_1$, because j_1 and $-j_1$ are the eigenvalues of $J_z^{(1)}$ with the largest modulus. As a consequence, we have for every product state

$$|\langle D^{(2)}\rangle| = |\langle\vec{J}^{(1)}\cdot\vec{J}^{(2)}\rangle| \,/ (j_1 j_2) \leq 1. \quad (2)$$

The bound of Eq. (2) can be extended straightforwardly to separable states given the following triangle inequality (the expectation value for product states has suffix ψ):

$$\left|\sum_\psi \mathrm{p}_\psi \langle\vec{J}^{(1)}\cdot\vec{J}^{(2)}\rangle_\psi\right| \;\leq\; \sum_\psi \mathrm{p}_\psi \left|\langle\vec{J}^{(1)}\cdot\vec{J}^{(2)}\rangle_\psi\right| \quad (3)$$

Note that the choice of co-ordinate system for each spin is arbitrary.

What is the maximal violation of Eq. (2) for entangled states? Noting the relation $(\vec{J}^{(1)}.\vec{J}^{(2)}) = \frac{1}{2}(\vec{J}^{\,2} - \vec{J}^{(1)\,2} - \vec{J}^{(2)\,2})$, with $\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$, indicates that the eigenstates of $(\vec{J}^{(1)}.\vec{J}^{(2)})$ are also eigenstates of $\vec{J}^{\,2}$, corresponding to $j(j+1)$ where $j = j_1 + j_2 - \lambda$, and $\lambda \in \mathbb{N}_0 \leq 2\mathrm{Min}[j_1, j_2]$. Assuming $j_1 \leq j_2$ and that subspaces (1) and (2) have fixed dimension gives:

$$-\left(1 + \frac{1}{j_2}\right) \leq \frac{1}{j_1 j_2}\left(\langle\vec{J}^{(1)}.\,\vec{J}^{(2)}\rangle\right) \leq 1 \quad (4)$$

From this result it is seen that the maximum absolute value is 3, provided by the 2-qubit singlet, $j_1 = j_2 = 1/2$.

Cross Product and Multipartite Inequalities: In full analogy with the dot product, the expectation value of the cross product in a product state is the cross product of the spin expectation value vectors for individual systems,

$$\langle\psi_{12}|\vec{J}^{(2)}\times\vec{J}^{(1)}|\psi_{12}\rangle = \langle\chi_2|\vec{J}^{(2)}|\chi_2\rangle\times\langle\phi_1|\vec{J}^{(1)}|\phi_1\rangle. \quad (5)$$

As the norm of the cross product of two vectors is bounded from above by the product of the their norms, we find for product states

$$||\langle\vec{C}^{(2)}\rangle|| = ||\langle\vec{J}^{(2)}\times\vec{J}^{(1)}\rangle|| \,/\, (j_1 j_2) \;\leq 1, \quad (6)$$

and again the generalization to separable states is immediate.

(N,J)	Max $\|\|\langle\vec{C}^{(N)}\rangle\|\|$	Max $\|\langle D^{(N)}\rangle\|$
$(2,1/2)$	2	3
$(3,1/2)$	$2\sqrt{2}$ (≈ 2.828)	$2\sqrt{3}$ (≈ 3.464)
$(4,1/2)$	$2\sqrt{6}$ (≈ 4.899)	$4\sqrt{3}$ (≈ 6.928)
$(5,1/2)$	$2\sqrt{14}$ (≈ 7.483)	$4\sqrt{6}$ (≈ 9.798)
$(6,1/2)$	≈ 12.144	≈ 16.971
$(2,1)$	$\sqrt{2}$ (≈ 1.414)	2
$(3,1)$	$\sqrt{3}$ (≈ 1.7321)	$\sqrt{3}$ (≈ 1.7321)
$(4,1)$	$\sqrt{3+\sqrt{5}}$ (≈ 2.288)	$2\sqrt{2}$ (≈ 2.828)
$(5,1)$	≈ 2.840	3
$(6,1)$	≈ 3.731	≈ 4.472

TABLE I: Some maximal violations for the inequalities of Eq. (7) and Eq. (9), for entangled states of N qubits ($J = 1/2$) and qutrits ($J = 1$). The largest violation we were able to find numerically was Max$\langle D^{(11)}\rangle \approx 152.691$, for 11 qubits. $D^{(11)}$ has 2^{11} eigenvalues with some degeneracy, see Eq. (11).

By iterating the cross product operation it is possible to derive bounds for multipartite systems. For a fully separable state of N spins $j_1, j_2, j_3, ..., j_N$ one derives the following bound, in analogy with Eq. (6):

$$||\langle\vec{C}^{(N)}\rangle|| \leq 1 \tag{7}$$

where we use the notation:

$$\vec{C}^{(N)} = \vec{J}^{(N)} \times (\vec{J}^{(N-1)} \times ...(\vec{J}^{(2)} \times J^{(1)}))/(j_1 j_2 ... j_N). \tag{8}$$

Considering the dot product between a single spin and a vector constructed like in Eq. (8) one can also derive a bound

$$|\langle D^{(N)}\rangle| \leq 1. \tag{9}$$

for fully separable states, where

$$D^{(N)} = \vec{J}^{(N)} \cdot (\vec{J}^{(N-1)} ... \times (\vec{J}^{(2)} \times J^{(1)}))/(j_1 j_2 ... j_N). \tag{10}$$

We have investigated how strongly the bounds of Eq. (7) and Eq. (9) can be violated by entangled states, TABLE I. For $D^{(N)}$ this involves finding its largest eigenvalue. Maximum values of $||\vec{C}^{(N)}||$ may be found by studying the greatest expectation value for any of the vector components of $\vec{C}^{(N)}$. This is because joint identical SU(2) transformations on all spins (i.e. each with the same $\vec{\alpha}$) correspond to simple rotations of the vector $\langle\vec{C}^{(N)}\rangle$. It can thus always be brought to a standard form where only one of its components e.g. $C_z^{(N)}$ is non-zero, without changing its norm. The maximum of the norm is therefore the largest eigenvalue of $C_z^{(N)}$. Upper bounds also exist for all entanglement partitions and one partition may have a range of upper bounds depending on the ordering of sub-systems $(1), (2), \ldots, (N)$ in the directed products $D^{(N)}$ and $\vec{C}^{(N)}$, see TABLE II.

Symmetry and Eigenstates: In addition to the numerical results, some insight into the structure of the eigenstates may be gained from symmetry considerations. $\langle D^{(N)}\rangle$ and $\langle\vec{C}^{(N)}\rangle$ transform like a scalar and a vector respectively under all identical joint rotations $R(\vec{\alpha})^{\otimes N}$, or equivalently under all identical local $SU(2)$ transformations of the state. Furthermore, $D^{(N)}$ is anti-symmetric under the permutation $(1) \leftrightarrow (2)$ because $(\vec{J}^{(1)} \times \vec{J}^{(2)}) = -(\vec{J}^{(2)} \times \vec{J}^{(1)})$. Operator $D^{(N)}$ must therefore have the following highly symmetric structure:

$$D^{(N)} = \sum_{\mathcal{D}} \mu_{\mathcal{D}} (\Pi_{\mathcal{D}}^{(12)} - \Pi_{\mathcal{D}}^{(21)}) \tag{11}$$

i.e. a weighted sum of projectors $\Pi_{\mathcal{D}}$ onto spaces associated with irreducible matrix representations $\mathcal{D}$ of $SU(2)$. All the projectors are orthogonal, $\Pi_{\mathcal{D}}\Pi_{\mathcal{D}'} = \delta_{\mathcal{D}\mathcal{D}'}\Pi_{\mathcal{D}}$ and they each project into a $(2J+1)$ dimensional space of overall spin $J(\mathcal{D})$. States in each representation $\mathcal{D}$ have a shared permutation symmetry of all the N particles; e.g. the highest spin representation $\mathcal{D}^*$, for which $J = j^{(1)} + \ldots + j^{(N)}$, is inclusive of all states symmetric under all particle permutations. The projector $\Pi^{(21)}$ is formed from $\Pi^{(12)}$ by exchanging particles labelled '(1)' and '(2)', mapping either $\Pi_{\mathcal{D}}^{(12)}$ to itself, in which case it vanishes from Eq. (11), or to an orthogonal projector of the same spin. Thus, eigenvalues of $D^{(N)}$ appear in pairs of opposite sign $\pm(\mu_{\mathcal{D}} - \mu_{\mathcal{D}'})$, with the multiplicity of pairs $2J(\mathcal{D}) + 1$. To give a concrete example, eigenstate $|\psi_4\rangle$ maximally violates $|\langle D^{(4)}\rangle| \leq 1$:

$$2\sqrt{6}\,|\psi_4\rangle = (1+\sqrt{3})(|\uparrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle)$$
$$+(1-\sqrt{3})(|\downarrow\uparrow\downarrow\uparrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle) - 2(|\downarrow\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle) \tag{12}$$

This state is one of two orthogonal $J = 0$ states of 4 qubits, and gives $D^{(4)}|\psi_4\rangle = 4\sqrt{3}|\psi_4\rangle$.

Since $\vec{C}^{(N)}$ is a vector operator i.e. $[C_u^{(N)}, J_v] = i\epsilon^{uvw}C_w^{(N)}$ for $u, v, w \in \{x, y, z\}$, it is a spin-1 object, of which $C_z^{(N)}$ is the $m = 0$ component. As a consequence and

Partition	Max$\|\|\langle\vec{C}^{(4)}\rangle\|\|$	Max$\|\langle D^{(4)}\rangle\|$	Max$\langle F^{(4)}\rangle$
[1 2 3 4]	$2\sqrt{6}$ (≈ 4.899)	$4\sqrt{3}$ (≈ 6.928)	$2\sqrt{2}$ (≈ 2.828)
[1 2 3|4]	$2\sqrt{2}$ (≈ 2.828)	$2\sqrt{2}$	2
[1 2|3 4]	4	6	$\sqrt{2}$
[1|2|3 4]	2	2	$\sqrt{2}$ (≈ 1.414)
[1|2|3|4]	1	1	1

TABLE II: Magnitudes of $\langle\vec{C}^{(4)}\rangle$ and $\langle D^{(4)}\rangle$ have distinct upper bounds (found numerically) for entanglement partitions of 4 qubits, labelled '(1)' to '(4)'. No entanglement exists between qubits separated by a vertical bar. There are 4! permutations for the directed products $\vec{C}^{(4)}$ and $D^{(4)}$, resulting in a range of upper bounds for some partitions, the largest of which is shown above. For example, [1|2|3 4] has three distinct bounds for Max$||\langle\vec{C}^{(4)}\rangle||$, namely 1, $\sqrt{2}$ and 2. Note that the Mermin-Klyshko operator $F^{(4)}$ gives identical upper bounds [5] for [1|2|34] and [12|34]; its expectation value is also degenerate under all particle re-orderings. The new inequalities produce a larger maximal violation. Because the new partition bounds are the result of a (global) numerical optimization, greater computing power and a more refined search may allow some of them to be improved.

in contrast to $D^{(N)}$, operator $C_z^{(N)}$ is not diagonal in the spin basis. The Wigner-Eckart theorem [11] may be invoked to reveal that

$$\langle J', m'|C_z^{(N)}|J, m\rangle = \langle J, 1; m, 0|J, 1; J', m'\rangle\, T_{J,J'} \quad (13)$$

where $\langle j_1, j_2; m_1, m_2|j_1, j_2; j, m\rangle$ are the Clebsch-Gordan coefficients and $T_{J,J'}$ is a transition matrix element dependent only on J and J'. The spin-1 selection rules are the familiar ones of dipole radiation; $(J' - J)$ and $(m' - m) \in \{-1, 0, 1\}$, with the $J = J' = 0$ transition forbidden, i.e. $T_{0,0} = 0$. From this perspective, the scalar operator $D^{(N)}$ is a spin-0 object, only able to couple states for $J = J'$ and $m = m'$.

The pair anti-symmetry of $D^{(N)}$ is also true of $C_z^{(N)}$ (eigenvalues appear in pairs $\pm\chi$). An example of the states maximally violating Eq. (7) is $|\phi_4\rangle$,

$$\begin{aligned} 4\sqrt{3}\,|\phi_4\rangle = 3(|\uparrow\downarrow\uparrow\uparrow\rangle - |\downarrow\uparrow\downarrow\downarrow\rangle) \\ +\sqrt{6}(|\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\downarrow\uparrow\rangle) \\ +|\downarrow\downarrow\downarrow\uparrow\rangle - |\uparrow\uparrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\downarrow\rangle - |\uparrow\uparrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\downarrow\rangle - |\downarrow\uparrow\uparrow\uparrow\rangle \end{aligned} \quad (14)$$

which gives $C_z^{(4)}|\phi_4\rangle = 2\sqrt{6}|\phi_4\rangle$, cf. TABLE II. This is one of four orthogonal states that give the same maximum.

Comparison with Mermin-Klyshko Inequalities: There are 2^{2^N} independent Bell inequalities for N qubit ensembles having two two-valued observables per qubit or 'site', [12]. This set is the simplest and best understood of multi-partite Bell inequalities, although others may be formulated with e.g. 3 observables per site, [13]. The 2^{2^N} inequalities are satisfied by all local hidden variable theories, and all are maximally violated by GHZ states:

$$|\mathrm{GHZ}_N\rangle = (\,|\uparrow\rangle^{\otimes N} + |\downarrow\rangle^{\otimes N}\,)/\sqrt{2}. \quad (15)$$

Note that the states which maximally violate Eq. (7) and Eq. (9), e.g. $|\phi_4\rangle$ and $|\psi_4\rangle$, are generally inequivalent to $|\mathrm{GHZ}_4\rangle$ under local unitaries. This can be proved by determining the Schmidt coefficients of the states for a bipartite $\{2, 2\}$ cut.

For the Bell inequalities of the type mentioned, the Mermin-Klyshko (MK) inequality [3] has the largest possible violation [12], within the context of quantum mechanics. The MK inequality depends on operator functions $F^{(N)}$ that may be defined recursively:

$$2F^{(N)} = F^{(N-1)} \otimes (A^{(N)} + \tilde{A}^{(N)}) + \tilde{F}^{(N-1)} \otimes (A^{(N)} - \tilde{A}^{(N)}) \quad (16)$$

where $A^{(N)}$ and $\tilde{A}^{(N)}$ are observables of the Nth qubit with eigenvalues ± 1, and $F^{(1)} = A^{(1)}$. Operator $\tilde{F}^{(N)}$ is obtained under exchange $A \leftrightarrow \tilde{A}$ for all the nested observables. The MK inequality satisfied by local hidden variable theories is $\langle F^{(N)}\rangle \leq 1$.

For any quantum state, $F^{(N)}$ has an upper bound, $\langle F^{(N)}\rangle \leq 2^{(N-1)/2}$, resulting in a smaller possible violation of the MK inequality above than that attainable for Eq. (7) and Eq. (9) in qubit ensembles $N \mapsto \{2, 3, 4, \ldots, 11\}$, cf. TABLE I. Also,

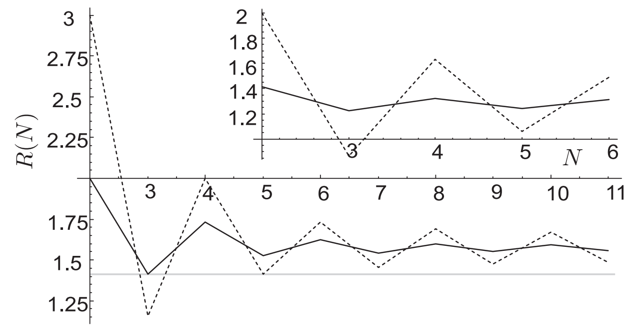

FIG. 1: Ratio of successive maximal violations for qubits and (inset) qutrits. $R(N) \mapsto \mathrm{Max}||\langle\vec{C}^{(N)}\rangle||/\mathrm{Max}||\langle\vec{C}^{(N-1)}\rangle||$ (unbroken line), and $\mapsto \mathrm{Max}|\langle D^{(N)}\rangle|/\mathrm{Max}|\langle D^{(N-1)}\rangle|$ (dashed line). $D^{(1)}$ is unity, and $\vec{C}^{(1)} = \vec{J}/j$. The ratio for Mermin-Klyshko inequalities is $\sqrt{2}$ (grey horizontal line).

the ratio of maximal violation for N qubits to $(N-1)$ qubits is always $\sqrt{2}$ for the MK inequalities, whereas for the inequalities of Eq. (7) the ratio is always $\geq \sqrt{2}$ but displays an intriguing 'see-saw' character; FIG.1 gives ratios for qubits and qutrits, and for both Dot and Cross inequalities. That a larger violation is possible for the new inequalities may be attributable at least in part to having three (orthogonal) observables per qubit, compared to two for the MK inequalities. Another possible reason is that our inequalities do not exclude the possibility of a local hidden variable model. They are strictly criteria for non-separability.

Operators $\vec{C}^{(N)}$ and $D^{(N)}$ are also unlike $F^{(N)}$ in that they are not symmetric under particle exchange; $\vec{C}^{(N)}$ is a 'directed' product. TABLE II shows how different particle orderings results in different bounds on $\vec{C}^{(4)}$ and $D^{(4)}$.

In terms of the numerical search needed to find a violation, three parameters will define 3 orthogonal directions in $\mathbb{R}^3$, whereas 4 variables are needed for both measurement directions per subspace in a Bell inequality. Thus, fewer parameters are required in the optimisation of Eq. (8) and Eq. (10).

Entanglement detection and Robustness: Consider $D^{(3)} = \vec{J}^{(3)}.(\vec{J}^{(2)} \times \vec{J}^{(1)})/j_1 j_2 j_3$ for 3 qubits. Its largest amplitude eigenvalue is $2\sqrt{3}$, an associated eigenstate is

$$|W_3\rangle = (\,|\uparrow\uparrow\downarrow\rangle + e^{i\alpha}|\uparrow\downarrow\uparrow\rangle + e^{i\beta}|\downarrow\uparrow\uparrow\rangle\,)/\sqrt{3}\,, \quad (17)$$

a state inequivalent to $|\mathrm{GHZ}_3\rangle$ under rotations of the local coordinate systems. For GHZ states, numerical optimisation over all local coordinate systems gives $|\langle D^{(3)}\rangle| \leq \frac{3}{2}\sqrt{3}$, i.e. the maximum possible violation is smaller than that for the W state by a factor of $3/4$. For completely separable states, $|\langle D^{(3)}\rangle|$ corresponds to the volume of a parallelopiped with sides of unit length. The detection of W-type entanglement is robust against noise: Mixed with white noise, $(1-\nu)|W_3\rangle\langle W_3| + \nu\,\mathbb{I}_8/8$, the fraction of noise ν can be as high as 71% and $|\langle D^{(3)}\rangle| \leq 1$ is still violated. Substituting $|\mathrm{GHZ}_3\rangle$, violation occurs for $\nu \leq 61\%$. The 3 qubit MK inequality will only detect $|\mathrm{GHZ}_3\rangle$ mixed with less than 50% $\mathbb{I}_8/8$, even though it is maximally violated by such states.

Violation Ratio: We now show that for a given entangled state the maximum possible violation for the Cross criterion, when optimizing the choice of local coordinate systems, cannot be larger than the maximum violation for the Dot criterion. Consider the correlation $i \in \{x, y, z\}$ defined as $i \equiv \langle J_i^{(N)} \otimes C_i^{(N-1)} \rangle / j^{(N)}$. One may write:

$$D^{(N)} = \sum_{i \in \{x,y,z\}} (J_i^{(N)} \otimes C_i^{(N-1)}) \Big/ j^{(N)} = x + y + z \quad (18)$$

A correlation set $\{x, y, z\}$ may be mapped into $\{x, -y, -z\}, \{-x, y, -z\}$ and $\{-x, -y, z\}$ by local unitaries (π rotations of the Nth qubit about x, y, z axes respectively). Therefore Max$|\langle D^{(N)} \rangle|$ is associated with correlations $\langle J_i^{(N)} \otimes C_i^{(N-1)} \rangle$ all having the *same* sign, i.e. Max$|\langle D^{(N)} \rangle| = $ Max$(|x| + |y| + |z|)$. For the Cross criterion, taking Max$||\langle \vec{C}^{(N)} \rangle|| = $ Max$|\langle C_z^{(N)} \rangle|$, one has:

$$\begin{aligned} \text{Max}|\langle C_z^{(N)} \rangle| &= \text{Max} \left| \left\langle \frac{J_y^{(N)}}{j^{(N)}} \otimes C_x^{(N-1)} - \frac{J_x^{(N)}}{j^{(N)}} \otimes C_y^{(N-1)} \right\rangle \right| \\ &= \text{Max} \left| \left\langle J_x^{(N)} \otimes C_x^{(N-1)} + J_y^{(N)} \otimes C_y^{(N-1)} \right\rangle \right| \Big/ j^{(N)} \quad (19) \end{aligned}$$

(which is Max$|x + y|$) because a local rotation in the Nth subspace transforms $\{J_x^{(N)}, J_y^{(N)}\} \mapsto \{-J_y^{(N)}, J_x^{(N)}\}$ above. Thus for a given state, the maximum of $|\langle C_z^{(N)} \rangle|$ for all choices of local coordinate systems is at most a sum of two of the absolute values, $|x|$, $|y|$ and $|z|$, obviously upper-bounded by Max$(|x| + |y| + |z|)$

$$\text{Max}||\langle \vec{C}^{(N)} \rangle|| \leq \text{Max}|\langle D^{(N)} \rangle| \quad (20)$$

It is stressed that for Eq. (20) the particles or subspaces are considered in the same order for both $\vec{C}^{(N)}$ and $D^{(N)}$.

Bound Entanglement: We consider a mixture of GHZ and product state projectors [6]:

$$\rho_N = \frac{1}{N+1} \left(|\text{GHZ}_N\rangle\langle\text{GHZ}_N| + \frac{1}{2} \sum_{n=1}^{N} (\Pi_n + \Pi_{\tilde{n}}) \right) \quad (21)$$

Here Π_n is a projector onto product state $|\uparrow\rangle_1 |\uparrow\rangle_2 \ldots |\downarrow\rangle_n \ldots |\uparrow\rangle_N$, i.e. only the nth qubit is in the orthogonal state. Projector $\Pi_{\tilde{n}}$ is obtained from Π_n by interchanging all $\uparrow$ with $\downarrow$. In [6] it was shown that ρ_N is entangled for $N \geq 4$ (it has negative partial transposition in some partitions) but entanglement cannot be distilled for any $\{1, (N-1)\}$ partition, it is 'bound' [15]. See also [7]. The state violates $\langle F^N \rangle \leq 1$ if and only if $N \geq 8$ and violates a Bell inequality of three dichotomic observables per qubit [13] for $N \geq 7$; recently it was shown to violate a 'functional' Bell inequality [16] for $N \geq 6$. Bell violations for lower N have yet to be shown. In contrast, the $N = 4$ state violates both 'Cross' and 'Dot' inequalities: A numerical search over local unitaries gives Max$||\langle \vec{C}^{(4)} \rangle|| \approx 1.09$ and Max$|\langle D^{(4)} \rangle| \approx 1.25$ for ρ_4. This result was unexpected, compared with the non-violation of MK inequalities by ρ_4. After all, the only entanglement in ρ_N is due to the GHZ state – this maximally violates the MK inequality.

Summary: A simple geometric approach allows the formulation of entanglement inequalities based on the scalar and vector products of two spin operators. This idea was extended to inequalities for multiple subsystems, each of arbitrary dimensionality. The maximum violation for N qubits is greater than that for certain classes of Bell inequalities, including the Mermin-Klyshko inequalities. The maximally violating states are not generally GHZ states; examples were given and elements of their structure discussed. $D^{(3)}$ showed a high level of robustness in detecting both W and GHZ entanglement. Maximal violations for all entanglement partitions of 4 qubits were found numerically – none of them is degenerate (unlike the MK bounds), and maximum violation is for a fully 4-entangled state. We have also shown that the new inequalities can detect bound entanglement in states for which a Bell violation has not been found. The work here followed a spin description, but may be applied to arbitrary systems for which $\{J_x, J_y, J_z\}$ operators can be defined and measured.

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[1] D.M. Greenberger, M.A. Horne, and A. Zeilinger, in M. Kafatos et al. (Eds.), *Bell's Theorem, Quantum Theory and Conceptions of the Universe* (Kluwer Academic, Dordrecht, 1989) .

[2] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2000).

[3] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990); A. V. Belinskii and D. N. Klyshko, Sov. Phys. Usp. **36**, 653 (1993).

[4] M. Zukowski and C. Brukner, Phys. Rev. Lett. **88**, 210401 (2002); J. Uffink, Phys. Rev. Lett. **88**, 230406 (2002); K. Nagata, M. Koashi, and N. Imoto, Phys. Rev. Lett. **89**, 260401 (2002); S. Yu *et al.*, Phys. Rev. Lett. **90**, 080401 (2003); W. Laskowski *et al.*, Phys. Rev. Lett. **93**, 200401 (2004).

[5] D. Collins *et al.*, Phys. Rev. Lett. **88**, 170405 (2002).

[6] W. Dür, Phys. Rev. Lett. **87**, 230402 (2001).

[7] A. Acín, Phys. Rev. Lett. **88**, 027901 (2002).

[8] T.-C. Wei *et al.*, Phys. Rev. A **70**, 022322 (2004), A. S. Sørensen and K. Mølmer, Phys. Rev. Lett. **86**, 4431 (2001); G. Tóth and O. Gühne, Phys. Rev. Lett. **94**, 060501 (2005).

[9] C. Simon and D. Bouwmeester, Phys. Rev. Lett. **91**, 053601 (2003).

[10] A. Sørensen *et al.*, Nature **409**, 63 (2001).

[11] J. J. Sakurai, *Modern Quantum Mechanics* (Addison Wesley Longman, 1994), chap. 3, revised ed.

[12] R. F. Werner and M. M. Wolf, Phys. Rev. A **64**, 032112 (2001).

[13] D. Kaszlikowski *et al.*, Phys. Rev. A **66**, 052309 (2002)

[14] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).

[15] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).

[16] A Sen (De), U. Sen, and M. Zukowski, Phys. Rev. A **66**, 062318 (2002).